3.1 Quadratic Functions

Functions

Let’s quickly review again the definition of a function.

Definition 1
A relation is a function if and only if each object in its domain is paired with one and only one object in its range.

Consider the relation $R$.

$$R = \{(0, 1), (0, 2), (3, 4)\}$$

The domain is \{0, 3\} and the range is \{1, 2, 4\}. Note that the number 0 in the domain of $R$ is paired with two numbers from the range, namely, 1 and 2. Therefore, $R$ is not a function.

There is a construct, called a mapping diagram, which can be helpful in determining whether a relation is a function. To craft a mapping diagram, first list the domain on the left, then the range on the right, then use arrows to indicate the ordered pairs in your relation, as shown in Figure 1.

$$\begin{array}{|c|c|}
\hline
0 & 1 \\
\hline
3 & 2 \\
\hline
3 & 4 \\
\hline
\end{array}$$

Figure 1. A mapping diagram for $R$.

It’s clear from the mapping diagram in Figure 1 that the number 0 in the domain is being paired (mapped) with two different range objects, namely, 1 and 2. Thus, $R$ is not a function.

Let’s look at another example.

Example 1

Is the relation described in $T$ a function?

First, the listing of the relation $T$.

$$T = \{(1, 2), (3, 2), (4, 5)\}$$

Next, construct a mapping diagram for the relation $T$. List the domain on the left, the range on the right, then use arrows to indicate the pairings, as shown in Figure 2.
Figure 2. A mapping diagram for $T$.

From the mapping diagram in Figure 2, we can see that each domain object on the left is paired (mapped) with exactly one range object on the right. Hence, the relation $T$ is a function.

Let’s summarize our review about functions.

**Summary 1**

A function consists of three parts:

1. a set of objects which mathematicians call the **domain**,  
2. a second set of objects which mathematicians call the **range**,  
3. and a **rule** that describes how to assign a unique range object to each object in the domain.

We now turn our attention to a particular type of function, the **quadratic function**

**The Quadratic Function**

There are several forms that a **quadratic** function can take on. The first we’ll review is called **Standard Form**.

$$f(x) = ax^2 + bx + c.$$  \hspace{1cm} (3.1)

Note that the coefficients $a$, $b$, and $c$ are real values.

Let’s now review how to draw the graph of the quadratic function. The form that is most useful for this task is known as **Vertex Form** and is given by the equation

$$f(x) = a(x - h)^2 + k.$$  \hspace{1cm} (3.2)

Recall that the graph of the quadratic function is shaped like a 'U' and is called a **parabola**. The form of the quadratic function in equation (3.2) is called vertex form, because the form easily reveals the vertex or “turning point” of the parabola. Each of the constants in the vertex form of the quadratic function plays a role. The constant $a$ controls the scaling (stretching or compressing of the parabola), the constant $h$ controls a horizontal shift and placement of the axis of symmetry, and the constant $k$ controls the vertical shift.
First let’s review what the scaling factor of the quadratic does to the graph.

**Property 1**

- If \( a \) is a constant larger than 1, that is, if \( a > 1 \), then the graph of \( g(x) = ax^2 \), when compared with the graph of \( f(x) = x^2 \), is stretched.
- If \( a \) is a constant smaller than 1 (but larger than zero), that is, if \( 0 < a < 1 \), then the graph of \( g(x) = ax^2 \), when compared with the graph of \( f(x) = x^2 \), is compressed.

Also, we need to keep in mind what happens when \( -1 < a < 0 \). In this case we get a *vertical reflection*. That is the parabola is reflected across the \( x \)-axis.

**Property 2**

- If \( -1 < a < 0 \), then the graph of \( g(x) = ax^2 \), when compared with the graph of \( f(x) = x^2 \), is compressed, then reflected across the \( x \)-axis.
- If \( a < -1 \), then the graph of \( g(x) = ax^2 \), when compared with the graph of \( f(x) = x^2 \), is stretched, then reflected across the \( x \)-axis.

Now that we have reviewed what the scaling factor \( a \) does to the graph of a quadratic function, let’s remind ourselves what the value of \( h \) does to the graph. Recall that the value of \( h \) translates our parabola \( h \) units to the left or the right of the origin.

**Property 3**

- If \( h > 0 \), then the graph of \( g(x) = (x - h)^2 \) is shifted \( h \) units to the right of the graph of \( f(x) = x^2 \).
- If \( h < 0 \), then the graph of \( g(x) = (x - h)^2 \) is shifted \( c \) units to the left of the graph of \( f(x) = x^2 \).

Now let’s look at the last value in \( f(x) = a(x - h)^2 + k \), namely \( k \). Recall that the value of \( k \) controls the amount of *vertical* translation.
Property 4

- If $k > 0$, the graph of $g(x) = x^2 + k$ is shifted $k$ units upward from the graph of $f(x) = x^2$.
- If $k < 0$, the graph of $g(x) = x^2 + k$ is shifted $k$ units downward from the graph of $f(x) = x^2$.

Vertex of a Parabola

Now that we have a quick review of what role each of the values in the vertex form $(f(x) = a(x - h)^2 + k)$ of a quadratic function plays. Notice that the vertex of the parabola will have coordinates given by $(h, k)$. Consider the following example.

Example 2

Find the vertex of the graph of $y = -(x + 2)^2 + 3$.

To do this we need to write the equation in vertex form.

$$y = a(x - h)^2 + k$$
$$y = -(x + 2)^2 + 3$$
$$y = -1(x - (-2))^2 + 3$$

By inspection we now see that $a = -1$, $h = -2$, and $k = 3$. This gives us that the coordinates of the vertex are at $(-2, 3)$ and that parabola has been reflected to open downward.

One last piece of the puzzle to review before we return to our example. Recall the definition of the axis of symmetry

The graph of $y = x^2$ is symmetric with respect to the $y$-axis. One half of the parabola is a mirror image of the other with respect to the $y$-axis. We say the $y$-axis is acting as the axis of symmetry. If we translate our parabola, then the axis of symmetry will also be translated. Let’s finish the example we started.

Example 3

Sketch the graph of $y = -(x + 2)^2 + 3$.

Analyze the equation $y = -(x + 2)^2 + 3$. The minus sign tells us that the parabola “opens downward.” The presence of $x + 2$ indicates a shift of 2 units to the left. Finally, adding the 3 will shift the graph 3 units upward. Thus, we have a parabola that “opens downward” with vertex at $(-2, 3)$. This is shown in Figure 3.
The axis of symmetry passes through the vertex $(-2, 3)$ in Figure 3 and has equation $x = -2$. Note that the right half of the parabola is a mirror image of its left half across this axis of symmetry. If we need more accuracy then we can use the axis of symmetry and a few well chosen points to do so.

- Start by plotting the vertex and axis of symmetry as shown in Figure 4(a).
- Next, compute two points on either side of the axis of symmetry. We choose $x = -1$ and $x = 0$ and compute the corresponding $y$-values using the equation $y = -(x + 2)^2 + 3$.

\[
\begin{array}{|c|c|}
\hline
x & y = -(x + 2)^2 + 3 \\
\hline
-1 & 2 \\
0 & -1 \\
\hline
\end{array}
\]

Plot the points from the table, as shown in Figure 4(b).
- Finally, plot the mirror images of these points across the axis of symmetry, as shown in Figure 4(c).

Figure 4. Using the axis of symmetry to establish accuracy.
The image in Figure 4(c) clearly contains enough information to complete the graph of the parabola having equation $y = -(x + 2)^2 + 3$ in Figure 5.

![Figure 5](image)

Figure 5. An accurate plot of $y = -(x + 2)^2 + 3$.

Let’s summarize what we’ve reviewed so far.

**Summary 2**

The form of the quadratic function

$$f(x) = a(x - h)^2 + k$$

is called **vertex form**. The graph of this quadratic function is a **parabola**.

1. The graph of the parabola opens upward if $a > 0$, downward if $a < 0$.
2. If the magnitude of $a$ is larger than 1, then the graph of the parabola is stretched by a factor of $a$. If the magnitude of $a$ is smaller than 1, then the graph of the parabola is compressed by a factor of $1/a$.
3. The parabola is translated $h$ units to the right if $h > 0$, and $h$ units to the left if $h < 0$.
4. The parabola is translated $k$ units upward if $k > 0$, and $k$ units downward if $k < 0$.
5. The coordinates of the vertex are $(h, k)$.
6. The axis of symmetry is a vertical line through the vertex whose equation is $x = h$.

**Domain and Range of a Quadratic Function**

Let’s review how to find the domain and range a quadratic function. Consider the quadratic function given by $f(x) = 2(x - 2)^2 - 3$. Recall that the domain is the set of “permissible $x$-values.” This means that the domain is all real numbers, which we can write as follows: Domain = $\mathbb{R}$ or Domain = $(-\infty, \infty)$. 

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You can also project each point on the graph of \( f(x) = 2(x-2)^2 - 3 \) onto the \( x \)-axis, as shown in Figure 6(a). If you do this, then the entire axis will “lie in shadow,” so once again, the domain is all real numbers.

![Figure 6](image.png)

**Figure 6.** Projecting to find (a) the domain and (b) the range.

To determine the range of the function \( f(x) = 2(x-2)^2 - 3 \), project each point on the graph of \( f \) onto the \( y \)-axis, as shown in Figure 6(b). On the \( y \)-axis, all points greater than or equal to \(-3\) “lie in shadow,” so the range is described with \( \text{Range} = \{ y : y \geq -3 \} = [-3, \infty) \).

The following summarizes how one finds the domain and range of a quadratic function that is in vertex form.

### Summary 3

The domain of the quadratic function

\[
f(x) = a(x - h)^2 + k,
\]

regardless of the values of the parameters \( a, h, \) and \( k, \) is the set of all real numbers, easily described with \( \mathbb{R} \) or \( (-\infty, \infty) \). On the other hand, the range depends upon the values of \( a \) and \( k \).

- If \( a > 0 \), then the parabola opens upward and has vertex at \((h, k)\). Consequently, the range will be
  \[
  [k, \infty) = \{ y : y \geq k \}.
  \]

- If \( a < 0 \), then the parabola opens downward and has vertex at \((h, k)\). Consequently, the range will be
  \[
  (-\infty, k] = \{ y : y \leq k \}.
  \]
How to Find the Vertex

It should now be clear that the vertex of the parabola plays a crucial role when working with quadratic functions. We also know that we can complete the square to find the coordinates of the vertex. However, recall there is a much faster way of finding the vertex.

**Vertex Formula.** Given the parabola represented by the quadratic function

\[ y = ax^2 + bx + c, \]

the \( x \)-coordinate of the vertex is given by the formula

\[ x_{\text{vertex}} = -\frac{b}{2a}. \]

The \( y \)-coordinate of the vertex is given by

\[ y_{\text{vertex}} = f\left(-\frac{b}{2a}\right). \]

Let’s look at an example.

▶ **Example 4**

Consider the parabola having equation

\[ f(x) = -2x^2 + 3x - 8. \]

Find the coordinates of the vertex and give the domain and range.

First, use the vertex formula to find the \( x \)-coordinate of the vertex.

\[ x_{\text{vertex}} = -\frac{b}{2a} = -\frac{3}{2(-2)} = \frac{3}{4}. \]

Next, substitute \( x = 3/4 \) to find the corresponding \( y \)-coordinate.

\[
\begin{align*}
  f\left(\frac{3}{4}\right) &= -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 8 \\
  &= -2\left(\frac{9}{16}\right) + \frac{9}{4} - 8 \\
  &= -\frac{9}{8} + \frac{18}{8} - \frac{64}{8} \\
  &= -\frac{55}{8}
\end{align*}
\]

Thus, the coordinates of the vertex are \((3/4, -55/8)\).

To find the domain we recall that for all parabolas, the domain is given by \((-\infty, \infty)\).

To find the range, we first need to note that this is a downward opening parabola \((a < 0)\). Then we use the \( y \) value of our vertex to determine that the range is given by \((-\infty, -55/8]\).